TENSOR COMPLETION VIA ADAPTIVE SAMPLING OF TENSOR FIBERS: APPLICATION TO EFFICIENT INDOOR RF FINGERPRINTING

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ABSTRACT

In this paper, we consider tensor completion under adaptive sampling of tensor (a multidimensional array) fibers. Tensor fibers or *tubes* are vectors obtained by fixing all but one index of the array. This sampling is in contrast to the cases considered so far where one performs an adaptive element-wise sampling. In this context we exploit a recently proposed algebraic framework to model tensor data [1] and model the underlying data as a tensor with low tensor tubal-rank. Under this model we then present an algorithm for adaptive sampling and recovery, which is shown to be nearly optimal in terms of sampling complexity. We apply this algorithm for robust estimation of RF fingerprints for accurate indoor localization. We show the performance on real and synthetic data sets. Compared to existing methods, that are primarily based on non-adaptive matrix completion methods, adaptive tensor completion achieves significantly better performance.

Index Terms – Data completion, Adaptive Sampling, Tensor algebra, RF fingerprinting

1. INTRODUCTION

Matrix and tensor completion from limited measurements has recently received a lot of attention with applications to many domains ranging from recommender systems [2], computer vision [3, 4] to seismic data interpolation [5]. Recently adaptivity in sampling has been shown to be very powerful for both matrix and tensor completion [11, 21]. So far these papers have focused on adaptive sampling of tensor elements. However in some cases, it is not feasible to perform element-wise sampling and one needs to sample an entire fiber of a tensor. For example, we consider the case of indoor localization where at each location the observer samples an entire vector of RSS signal strengths from a given set of access points. In this case it is not immediately clear how to adaptively acquire data, using current methods, for a robust estimation of the fingerprints for all the locations. In this paper we consider such a scenario.

In this context we exploit a recently proposed tensor algebraic framework [1], which models 3D tensors as linear operators. In this framework one can obtain an SVD like factorization, referred to as the tensor-SVD (t-SVD), which is a rank-revealing factorization with extensions of the notion of column/row subspaces of a matrix to that of tensor-column/tensor-row subspaces. Under this algebraic framework we propose an adaptive tensor-column subspace estimation algorithm, which in essence is similar to the adaptive column-subspace estimation algorithm proposed in [11]. Under some incoherency conditions we show that the proposed method is nearly optimal in terms of sampling complexity.

We show the performance of the proposed method for an RF fingerprinting application, where the objective is to fill in 3D RF fingerprint data by adaptive spatial sampling of the received signal strength (RSS) from a set of access points (AP). For this application, our work is very different from existing approaches [7, 8, 13] that utilize *non-adaptive* matrix completion methods.

2. ALGEBRAIC FRAMEWORK FOR 3D TENSORS

Our approach rests on the algebraic framework developed and used in [1, 4, 10]. **Note** that this framework is different from the traditional *multilinear algebraic framework* for tensor decompositions [9] that has been considered so far in the literature for problems of completing multidimensional arrays [22–24] with different notions for tensor rank.

Notation- A third-order tensor is represented by calligraphic letters, denoted as $\mathcal{T} \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, and its (i, j, k)-th entry is $\mathcal{T}(i, j, k)$. A *tube* (or fiber) of a tensor is a 1-D section defined by fixing all indices but one. For example, in the RF fingerprinting application, we use tube $\mathcal{T}(i, j, :)$ to denote a fingerprint at a spatial reference point index (i, j). Similarly, a *slice* of a tensor is a 2-D section defined by fixing all but two indices. *frontal, lateral, horizontal slices* are denoted as $\mathcal{T}(:, :, k), \mathcal{T}(:, j, :), \mathcal{T}(i, :, :)$, respectively.

For two tubes $a, b \in \mathbb{R}^{1\times 1\times N_3}$, $a * b$ denotes the *circular convolution* between these two vectors. The algebraic development in [1] rests on defining a tensor-tensor product between two 3-D tensors, referred to as the t-product and uses circular convolution between tubes, as defined below.

Definition 1. ***t-product***. *The t-product* $\mathcal{C} = \mathcal{A} * \mathcal{B}$ *of* $\mathcal{A} \in \mathbb{R}^{N_1\times N_2\times N_3}$ *and* $\mathcal{B} \in \mathbb{R}^{N_2\times N_4\times N_3}$ *is a tensor of size* $N_1 \times N_4 \times N_3$ *whose* (i, j)*-th tube* $\mathcal{C}(i, j, :)$ *is given by* $\mathcal{C}(i, j, :) = \sum_{k=1}^{N_2} \mathcal{A}(i, k, :) * \mathcal{B}(k, j, :)$, *for* $i = 1, 2, ..., N_1$ *and* $j = 1, 2, ..., N_4$.

Remark *- A third-order tensor of size* $N_1 \times N_2 \times N_3$ *can be viewed as an* $N_1 \times N_2$ *matrix of tubes that are in the third-*

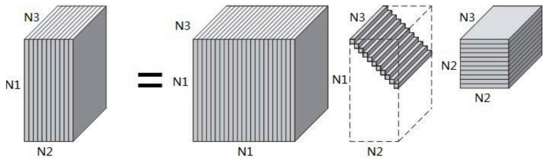

Fig. 1. The t-SVD of an $N_1 \times N_2 \times N_3$ tensor.

dimension. So the t-product of two tensors can be regarded as multiplication of two matrices, except that the multiplication of two numbers is replaced by the circular convolution of two tubes (tensor fibers).

This insight allows one to treat 3-D tensors as *linear operators* over 2-D matrices as analyzed in [1]. Using this perspective one can define a SVD type decomposition, referred to as the tensor-SVD (t-SVD). To define the t-SVD we introduce a few definitions.

Definition 2. ***Identity tensor***. *The identity tensor* $\mathcal{I} \in \mathbb{R}^{N_1 \times N_1 \times N_3}$ *is a tensor whose first frontal slice* $\mathcal{I}(:,:,1)$ *is the* $N_1 \times N_1$ *identity matrix and all other frontal slices are zero.*

Definition 3. ***Tensor transpose***. *The transpose of tensor* $\mathcal{T}$ *is the* $N_2 \times N_1 \times N_3$ *tensor* $\mathcal{T}^\top$ *obtained by transposing each of the frontal slices and then reversing the order of transposed frontal slices* 2 *through* N_3, *i.e., for* $k = 2, 3, ..., N_3$, $\mathcal{T}^\top(:,:,k) = (\mathcal{T}(:,:,N_3 + 2 - k))^\top$ *(the transpose of matrix* $\mathcal{T}(:,:,N_3 + 2 - k)$*).*

Definition 4. ***Orthogonal tensor***. *A tensor* $\mathcal{Q} \in \mathbb{R}^{N_1 \times N_1 \times N_3}$ *is orthogonal if it satisfies* $\mathcal{Q}^\top * \mathcal{Q} = \mathcal{Q} * \mathcal{Q}^\top = \mathcal{I}$.

Definition 5. ***f-diagonal tensor***. *A tensor is called f-diagonal if each frontal slice of the tensor is a diagonal matrix, i.e.,* $\mathcal{T}(i, j, k) = 0$ *for* $i \neq j, \forall k$.

Using these definitions one can obtain the t-SVD defined in the following result from [1]. Please see Figure 1 for a graphical representation.

Theorem 1. ***t-SVD***. *A tensor* $\mathcal{T} \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, *can be decomposed as* $\mathcal{T} = \mathcal{U} * \Theta * \mathcal{V}^\top$, *where* $\mathcal{U}$ *and* $\mathcal{V}$ *are orthogonal tensors of sizes* $N_1 \times N_1 \times N_3$ *and* $N_2 \times N_2 \times N_3$ *respectively, i.e.* $\mathcal{U}^\top * \mathcal{U} = \mathcal{I}$ *and* $\mathcal{V}^\top * \mathcal{V} = \mathcal{I}$ *and* Θ *is a rectangular f-diagonal tensor of size* $N_1 \times N_2 \times N_3$.

Definition 6. ***Tensor tubal-rank***. *The tensor tubal-rank of a third-order tensor is the number of non-zero fibers of* Θ *in the t-SVD.*

In this framework, the principle of dimensionality reduction follows from the following result from [1] [1].

[1]Note that Θ in t-SVD is organized in a decreasing order, i.e., $||\Theta(1,1,:)||_F \geq ||\Theta(2,2,:)||_F \geq ...$, which is implicitly defined in [4] as the algorithm for computing t-SVD is based on matrix SVD. Therefore, the best rank-r approximation of tensors is similar to PCA (principal component analysis).

Lemma 1. ***Best rank-r approximation***. *Let the t-SVD of* $\mathcal{T} \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ *be given by* $\mathcal{T} = \mathcal{U} * \Theta * \mathcal{V}^\top$ *and for* $r \leq \min(N_1, N_2)$ *define* $\mathcal{T}_r = \sum_{i=1}^{r} \mathcal{U}(:, i, :) * \Theta(i, i, :) * \mathcal{V}^\top(:, i, :)$, *then* $\mathcal{T}_r = \arg \min_{\overline{\mathcal{T}} \in \mathbb{T}} ||\mathcal{T} - \overline{\mathcal{T}}||_F$, *where* $\mathbb{T} = \{\Im = \mathcal{X} * \mathcal{Y} | \mathcal{X} \in \mathbb{R}^{N_1 \times r \times N_3}, \mathcal{Y} \in \mathbb{R}^{r \times N_2 \times N_3}\}$.

We now define the notion of ***tensor-column space*** - Under the proposed framework, a tensor-column subspace of a 3D tensor $\mathcal{T}$ is the space spanned by the lateral slices of $\mathcal{U}$ under the t-product, i.e., the set generated by t-linear combinations like so,

$$\text{t-colspan}(\mathcal{U}) = \{\mathbf{X} = \sum_{j=1}^{r} \mathcal{U}(:, j, :) * \mathbf{c}_j \in \mathbb{R}^{N_1 \times 1 \times N_3} \ , \mathbf{c}_j \in \mathbb{R}^{N_3}\},$$

where r denotes the tensor tubal-rank.

3. ADAPTIVE TENSOR COMPLETION

We consider a three-dimensional tensor of size $N_1 \times N_2 \times N_3$, where $N_1 N_2$ locations have N_3 features each, e.g. corresponding to RSS values from N_3 access points. Let $M \leq N_1 N_2$ denote the sampling budget, i.e., we can sample the tensor fibers along the third dimension at M reference points.

We model the *partial observation model* under ***tubal-sampling*** as follows:

$$\mathcal{Y} = \mathcal{P}_\Omega(\mathcal{T}) + \mathcal{P}_\Omega(\mathcal{N}), \ \ \Omega \subset \mathcal{G}, \tag{1}$$

where the (i, j, k)-th entry of $\mathcal{P}_\Omega(\mathcal{X})$ is equal to $\mathcal{X}(i, j, k)$ if $(i, j) \in \Omega$ and zero otherwise, i.e. *we sample the entire tensor fiber* $\mathcal{T}(i, j, :)$, Ω being a subset of the locations $\mathcal{G}$ and of size M, and $\mathcal{N}$ is an $N_1 \times N_2 \times N_3$ tensor with i.i.d. $N(0, \sigma^2)$ elements, representing the additive Gaussian noise. To cut down the survey burden, we measure the feature values of a small subset of reference points and then estimate $\mathcal{T}$ from the samples $\mathcal{Y}$. Under the assumption that tensor $\mathcal{T}$ has low-tubal-rank and given a sampling budget M, we want to solve the following optimization problem.

$$\langle \hat{\mathcal{T}}, \hat{\Omega} \rangle = \arg \min_{\mathcal{X}, \Omega} \ ||\mathcal{P}_\Omega(\mathcal{Y} - \mathcal{X})||_F^2 + \lambda \cdot \text{rank}(\mathcal{X}), : \ |\Omega| \leq M, \tag{2}$$

where $\mathcal{X}$ is the decision variable, rank$(\cdot)$ refers to the tensor tubal-rank, M is the sampling budget, and λ is a regularization parameter.

Note - *We want to emphasize again that the we are interested in sampling tensor fibers and not element-wise sampling as is the focus of many recent papers.*

Tensor completion with adaptive tubal-sampling - The optimization problem of Equation (2) contains two goals: (1) For a given low-tubal-rank tensor $\mathcal{X}$, to select a set Ω with the smallest cardinality and the corresponding samples $\mathcal{Y}$, preserving most information of tensor $\mathcal{X}$, i.e., one can recover $\mathcal{X}$ from Ω and $\mathcal{Y}$. (2) For a given set Ω and samples $\mathcal{Y}$, to estimate a tensor $\mathcal{X}$ that has the least tubal-rank. However, these two goals are intertwined together and one cannot expect a computationally feasible algorithm to get the optimal

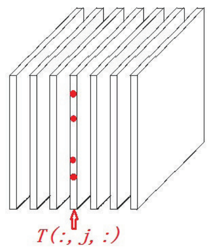

Fig. 2. A 3D tensor with its lateral slices. The lateral slices can be seen as column vectors, with tubal elements. The engineer performs site survey at a few random selected reference points (with red color) of the j-th tensor-column of $\mathcal{T}$. Knowing the r-dimensional tensor-column subspace $\mathcal{U}$, one can recover the j-th lateral slice $\mathcal{T}(:, j, :)$.

solution. Therefore, we set $|\Omega| = M$ and seek to select a set Ω and the corresponding samples $\mathcal{Y}$ that span the low-dimensional tensor-column subspace of $\mathcal{T}$. The focus of this section is to design an efficient sampling scheme and to provide a bound on the sampling budget M for reliable recovery.

To achieve this, we design a two-pass sampling scheme inspired by [15]. The pseudo-code of our adaptive sampling approach is shown in Algorithm 1. The inputs include the grid map $\mathcal{G}$, the sampling budget M, the size of the tensor, N_1, N_2, N_3, the allocation ratio δ, and the number of iterations L. The algorithm consists of three steps. The 1st-pass sampling is a uniform tubal-sampling, while the 2nd-pass sampling outputs an estimate $\hat{\mathcal{U}}$ of the tensor-column subspace $\mathcal{U}$ in L rounds, as explained below.

Algorithm 1 Tensor completion based on adaptive sampling

Input: parameters $\mathcal{G}, M, N_1, N_2, N_3, \delta, L$

1st-pass sampling:

Uniformly sample $\delta M/N_2$ reference points from each column of the grip map $\mathcal{G}$, denoted as Ω_j^1. $\Omega^1 = \bigcup_{j=1}^{N_2} \Omega_j^1$.

2nd-pass sampling:

$\hat{\mathcal{U}} \leftarrow \emptyset, \Pi \leftarrow \{1, 2, ..., N_2\}$.

for $l = 1 : L$ **do**

  Estimate $\hat{p}_j = \frac{||\mathcal{P}_{\hat{\mathcal{U}}^\perp}(\mathcal{T}(:,\Omega_j^1,:))||_F^2}{||\mathcal{P}_{\hat{\mathcal{U}}^\perp}(\mathcal{T}(:,\Omega^1,:))||_F^2}, \forall j \in \Pi$.

  Sample $s = \frac{(1-\delta)M}{N_2 L}$ columns of $\mathcal{G}$ according to $\hat{p}_j$ for $\forall j \in \Pi$, denoted as Π_s^l.

  Calculate $\mathbb{U} \leftarrow \frac{\mathcal{P}_{\hat{\mathcal{U}}^\perp}(\mathcal{T}(:,\Pi_s^l,:))}{||\mathcal{P}_{\hat{\mathcal{U}}^\perp}(\mathcal{T}(:,\Pi_s^l,:))||_F}$, and update $\hat{\mathcal{U}} \leftarrow \hat{\mathcal{U}} \cup \mathbb{U}$ (concatenate $\mathbb{U}$ to $\hat{\mathcal{U}}$).

  $\Pi \leftarrow \Pi / \Pi_s^l$ (set subtraction).

end for

Estimate $\hat{\mathcal{T}}(:, j, :) = \hat{\mathcal{U}} * (\hat{\mathcal{U}}_{\Omega_j}^T * \hat{\mathcal{U}}_{\Omega_j})^{-1} * \hat{\mathcal{U}}_{\Omega_j}^T * \mathcal{T}(\Omega_j, j, :)$.

In particular the total sampling budget $M (< N_1 N_2)$ is divided into δM and $(1-\delta)M$ for these two sampling passes and δ is called *the allocation ratio*. In the 1st-pass sampling, we randomly sample $\delta M/N_2$ out of N_1 reference points in each column of $\mathcal{G}$. In the 2nd-pass sampling, the remaining $(1-\delta)M$ samples are allocated to those highly informative columns identified by the 1st-pass sampling. Finally, tensor completion on those M RF fingerprints is performed to rebuild a fingerprint database. The provable optimality of this scheme rests on these three assumptions about $\mathcal{T}$.

- $\mathcal{T}$ is embedded in an r-dimensional tensor-column subspace $\mathcal{U}$, $r \ll \min(N_1, N_2)$;
- Learning $\mathcal{U}$ requires to know only r *linearly independent* lateral slices;[2]
- Knowing $\mathcal{U}$, randomly sampling a few tubes of the j-th column is enough to recover the lateral slice $\mathcal{T}(:, j, :)$;
- One can then concatenate all estimated lateral slices to form an estimated tensor.

However, we do not know the value of r *a priori* nor the linearity between any two lateral slices. Ideally, this problem can be solved by sampling each column according to the probability distribution where the probability p_j of sampling the j-th lateral slice is proportional to $||\mathcal{P}_{\mathcal{U}^\perp}\mathcal{T}(:, j, :)||_F^2$, i.e, $p_j = \frac{||\mathcal{P}_{\mathcal{U}^\perp}\mathcal{T}(:,j,:)||_F^2}{||\mathcal{P}_{\mathcal{U}^\perp}\mathcal{T}||_F^2}$. Updating the estimate of $\mathcal{U}$ iteratively, when cr ($c > 1$ is a small constant) columns are sampled, we can expect that with high probability, $||\mathcal{P}_{\mathcal{U}^\perp}\mathcal{T}(:, j, :)||_F^2 = 0$, $\forall j$. Note that $\mathcal{P}_{\mathcal{U}^\perp}(\cdot)$ denotes projection onto the orthogonal space of $\mathcal{U}$; in t-product form, $\mathcal{P}_{\mathcal{U}} = \mathcal{U} * (\mathcal{U}^T * \mathcal{U})^{-1} * \mathcal{U}^T$, $\mathcal{P}_{\mathcal{U}^\perp} = \mathcal{I} - \mathcal{P}_{\mathcal{U}}$ [1].

The challenge is that we cannot have the exact sampling probability p_j without sampling all reference points of the grid map $\mathcal{G}$. Exploiting the spatial correlation, one can estimate the sampling probability from missing data (sub-sampled data), as $\hat{p}_j = \frac{||\mathcal{P}_{\mathcal{U}^\perp}(\mathcal{T}(:,\Omega_j^1,:))||_F^2}{||\mathcal{P}_{\mathcal{U}^\perp}(\mathcal{T}(:,\Omega^1,:))||_F^2}$ in Algorithm 1. Under some conditions we can prove that $\hat{p}_j$ is a good estimation of p_j and hence the algorithm performs as expected.

Performance bounds - The results and conditions with proof for the success of the algorithm are outlined in the paper [25]. Essentially the algorithm succeeds with high probability if the number of measurements $M > \mathcal{O}(rN \log^2 N)$ where $N = \max\{N_1, N_2\}$ and if the energy is not concentrated on a few *horizontal slices* of the tensor. This condition is referred to as the tensor incoherency condition. In contrast to the incoherency conditions on both the lateral and the horizontal slices as required in [10] under random elementw-wise non-adaptive sampling, we only require the incoherency of the lateral slices when using the proposed adaptive tubal-sampling scheme.

4. PERFORMANCE EVALUATION - RF FINGERPRINT DATA COMPLETION

We select a region of 47.5m $\times$ 59.7m in a real office building, as shown in Fig. 3. It is divided into a 476 $\times$ 598 grid

[2]Note: A collection of r lateral slices $\mathcal{U}(:, j, :), j = 1, ..., r$ are said to be linearly independent (in the proposed setting) if $\sum_{j=1}^{r} \mathcal{U}(:, j, :) * \mathbf{c}_j = 0 \implies \mathbf{c}_j = 0, \forall j$.

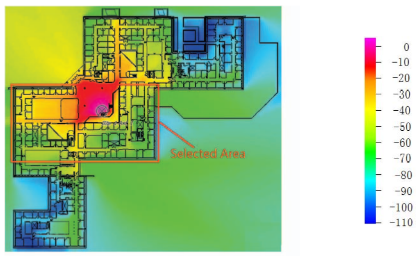

Fig. 3. RSS map for simulations measured in dBm.

map. There are 15 access points randomly deployed within this region. The ray tracing model [16, 17] is adopted, modeling all the propagation effects and a $476 \times 598 \times 15$ tensor is obtained as the *ground truth* containing the received signal strength (RSS) values from different access points, measured in dBm.

Varying the spatial sampling rate from 10% to 90%, we quantify the recovery error in terms of *normalized square of error (NSE)* for entries that are not sampled, i.e., recovery error for set Ω^c. The NSE is defined as:

$$NSE = \frac{\sum_{(i,j)\in\Omega^c} ||\hat{\mathcal{T}}(i,j,:) - \mathcal{T}(i,j,:)||_F^2}{\sum_{(i,j)\in\Omega^c} ||\mathcal{T}(i,j,:)||_F^2}, \qquad (3)$$

where $\hat{\mathcal{T}}$ is the estimated tensor, Ω^c is the complement of Ω.

For comparison, we consider three algorithms, tensor completion (TC) *under uniformly random tubal-sampling* and using the algorithm proposed in [4, 10], using the *face-wise* matrix completion (MC) algorithm in [18], and tensor completion via matricization or flattening (MC-flat) [24] under *uniform element-wise sampling* of the 3D tensor, using the AltMin algorithm for matrix completion [6].

Fig. 4 shows the RSS tensor recovery performance for varying sampling rate. Compared schemes are matrix completion and tensor completion via uniform sampling, and adaptive sampling with allocation ratio $\delta = 1/4$ and $\delta = 1/2$. We find that all tensor approaches are better than matrix completion, this is because tensor exploits the cross correlations among access points while matrix completion only takes advantage of correlation within each access point. Both AS schemes outperform tensor completion via uniform sampling since adaptivity can guide the sampling process to concentrate on more informative entries. Allocating equal sampling budget for the 1st-pass and the 2nd-pass gives better performance than uneven allocation. This shows that the 1st-pass and the 2nd-pass have equal importance. The proposed scheme (AS with $\delta = 1/2$) rebuilds a fingerprint data with 5% error using less than 30% samples.

We collected a WiFi RSS data set in the same office. The data set contains 89 selected locations and 31 access points. Since the locations are not exactly on a grid, we set the grid size to be 3m $\times$ 5m, and apply the k-nearest neighbor (KNN) method to extract a full third-order tensor as the ground truth.

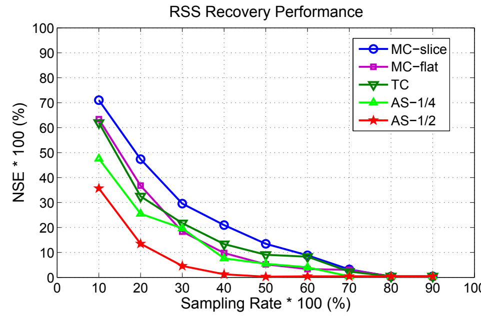

Fig. 4. Tensor recovery for varying sampling rate. MC denotes matrix completion via uniform sampling, TC denotes tensor completion via uniform sampling, while AS-$1/4$ and AS-$1/2$ denote our adaptive sampling scheme with allocation ratios $\delta = 1/4$ and $\delta = 1/2$, respectively. MC-flat denotes completion by *flattening* the 3-D data into a matrix of locations $\times$ APs and using matrix completion, an approach followed in [12].

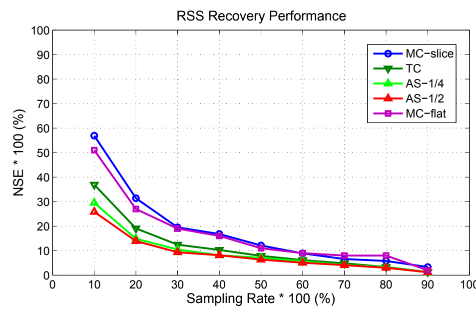

Fig. 5. Tensor recovery in the real-world data set. The same legend as in Fig. 4

To be specific, for each grid point, we set its RSS vector by averaging the RSS vectors from the nearest three ($k = 3$) locations. The ground truth tensor has dimension $10 \times 10 \times 31$.

Fig. 5 shows the RSS tensor recovery performance for the real-world data set. First, as compared with Fig. 4, we see that the recovery performance on real-world data is inconsistent with that of simulated data. Second, for real-world data set, tensor model is superior to matrix model. The reason is tensor can better exploit the spatial correlations of RSS values across multiple access points. In our case, a major ingredient for the recovery improvement may be the large number of access points (i.e., 31), compared with the dimension of the grid (i.e., $10x10$). Third, as expected, the propose adaptive scheme achieves better recovery performance.

5. CONCLUSIONS

In this paper, an adaptive sampling approach is proposed to relieve the number of locations at which feature vectors are measured. For low tubal-rank tensors, we proposed an algorithm for adaptive sampling. We show that the proposed scheme achieves near-optimal sampling complexity.

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