Upper Bound of The Number of Channels for Conflict-Free Communication in Multi-Channel Wireless Networks

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***Abstract*— In this paper, we address the following problem: given a set of wireless nodes on the plane, what is the number of channels needed to ensure a schedulable conflict-free communication? While the exact number of channels needed is very difficult to compute, we focus on an upper bound on this number, in terms of the node density D. In particular, we derive several upper bounds in terms of D based on different settings of network interference ratio r, and analyze the trend of the upper bound as r changes. We also examine the tightness of our upper bound, by studying some specific transmission scenarios.**

Our result can be viewed as another theoretic investigation towards the relationship of network performance and available number of channels. In practical application, the result suggests an easy estimation on the number of channels needed in the configuration of a wireless network. On the other hand, when the number of channels in a wireless system is given, the result can be used to provide a suggestion on the node density in the node deployment.

I. Introduction

The IEEE 802.11 standard [1] for wireless LAN offers multiple non-overlapping channels for use, to increase the available network capacity. It is recognized that by exploiting multiple channels, people can reduce conflict and utilize more resources, thus achieving a higher network throughput than using a single channel. Several MAC and routing protocols have been proposed for utilizing multiple channels in IEEE 802.11 [14] [9].

In a multi-channel wireless network, the relationship between the number of available channels and the network capacity has received interest from researchers [13] [10]. In this paper, we address this important problem from another viewpoint: how many channels are needed to ensure a schedulable conflict-free communication? Given a set of wireless nodes on the plain, it is very difficult to compute the exact number of channels needed. We try to search for upper bounds to this problem.

It is obvious that the number of channels needed depends heavily on the density of the wireless nodes. For a set of wireless nodes, we first define a measure of density D. This measure of density can be easily computed when given the locations of the nodes. We then adopt the widely-used model of *conflict graph*, and the problem of assigning channels to different communication links is converted to a coloring problem on the corresponding conflict graph. We show that the conflict graph resulting from the set of communication links in our model has some particular properties. By exploiting those properties, we derive upper bounds on the number of channels needed, in terms of D. In particular, we derive several upper bounds in terms of D based on different settings of network interference ratio r, and analyze the trend of the upper bound as r changes.

We also examine the tightness of our upper bound. By constructing and studying some specific patterns of node placement and transmission scenarios, we show a lower constraint of the upper bound. This gives us knowledge on how the upper bound is possible to be further improved.

Besides its theoretic contribution, our result is useful in practice. It can be used to estimate easily the number of channels needed in the configuration of a wireless network. On the other hand, when the number of channels in a wireless system is given, the result can be used to provide a suggestion on the node density or transmission power in the node deployment.

The remainder of this paper is organized as follows. In section II we provide a brief review of related work on wireless channel assignment and network capacity. In section III we define our channel assignment model and recall existing results on graph coloring. In section IV we present our upper bounds on the number of channels needed to ensure a conflict-free communication. In section V we construct and analyze some transmission scenarios and examine the tightness of our bound. Section VI summarizes this paper and discusses future work.

II. Related Work

There are mainly two categories of work related to this paper.

The relationship between the number of available channels and the network capacity has been studied by several researchers. In [13], the authors discussed the impact of number of channels and interfaces to the order of network capacity. In [10], the authors modelled the above problem using *linear programming*, and defined the constraints by

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interference constraint and *flow constraint*. They solved the model as an optimization problem through some heuristics, and derived upper bound and lower bound on the network capacity. Our work is a study of this interesting problem from another viewpoint: given a specific requirement on network performance (*i.e.* conflict-free communication), what is the number of channels needed?

Another set of research related to this paper are wireless link scheduling and channel assignment. There are a lot of work on finding optimal assignment and scheduling for a given network configuration. In [8], the authors proposed to use *conflict graph* to model the constraints, and the channel assignment problem was converted into a graph coloring problem. In [10] and [2], *Linear Programming* was adopted to model the problem as an optimization problem. In most cases, the optimization problems are NP-hard [16], and people have created efficient heuristics towards these problems [3], [11], [12]. While these work focus mainly on finding optimal assignments for a given network, we try to find general bounds on the number of channels needed, for any network with density D.

III. Background on Channel Assignment

The problem addressed in this paper is about link channel assignment, and is converted to a graph coloring problem by using conflict graphs. In the following, we start with our channel assignment model and formulate the problem of finding upper bounds, and briefly review the concept of conflict graph. Then we recall some preliminaries on graph coloring, which are used in the rest of the paper.

A. Problem Formulation

1) Basic Definition and Terminology: Consider a wireless network with N nodes located on a plane (we also use N to denote the set of nodes). Any two nodes may communicate and establish a *link* if they are within the *transmission range*, denoted r_t. The link established between node P and Q is simply denoted PQ. Two nodes *conflict* each other iff they are within the *interference range*, denoted r_i. r_t and r_i are fixed parameters in our model. Two links *conflict* with each other if either node of one link conflicts with either node of the other link.

Our model is similar to the *Protocol Model* introduced in [8] and [6].

Define the *interference ratio*: $r = r_i/r_t$. It is assumed that the interference range is not less than the transmission range, *i.e.* $r \geq 1$.

2) Link Set: A valid *link set* is defined as a set of links, such that no two links in the set share common nodes. Intuitively, a link set describes a set of links that needs to act simultaneously. That is, we assume a node can only communicate with at most one neighbor at one time. Note that it is not required that each node should appear in some link. We use LS to denote a link set.

3) Valid Assignment: A valid *assignment* to a link set is to assign each link a channel, such that no two conflicting links are assigned a same channel. A link set is said to be *schedulable* (using x channels) if there exists a valid assignment for the link set (using x channels).

4) Upper Bound Problem: Given a set of nodes N located on the plane, there are possibly many valid link sets. These different link sets represent the different combinations of communication pairs among the nodes. Our problem is to find a number, denoted U, such that any link set LS derived from N is schedulable using U channels. In other words, U is the upper bound of channels needed to ensure a conflict-free link assignment.

It is obvious that the number of U depends heavily on the number of nodes, as well as their placement. To quantify this property, we define a measure of *node density*, denoted D, as follows:

5) Node Density: Let C denote the (infinite) set of circles on the plane with radius r_i. Let $NC(c)$ denote the number of nodes in circle c. The density of nodes is defined as $D = \max_{c \in C} NC(c)$.

For practical use of our results, it is important that the density can be easily computed given the location of the nodes. Indeed there exists algorithms that can quickly compute D in $O(N^2)$ time [4]. In [4], the authors proposed to convert the problem of computing D into a so-called maximum weighted clique problem for circle intersection graphs, then apply a greedy algorithm to find D.

Our goal is to find the upper bound U, as tight as possible, in terms of node density D.

B. Conflict Graph

To address the link channel assignment process involved in our problem, we adopt the concept of *conflict graph* and convert the process into a graph coloring process.

Conflict graph is a useful model first proposed in [8]. The conflict graph for a given set of links is defined as follows. For every link, there is a vertex in the conflict graph representing that link. Two vertices in the conflict graph are connected by an edge if and only if the two links conflict. The conflict graph G constructed from a link set LS is denoted $G(LS)$.

By constructing the conflict graph for a link set, and representing each wireless channel using a different color, the requirement that "no two conflicting links share the same channel" in the link channel assignment is equivalent to the constraint that "no two adjacent nodes share the same color" in the graph coloring. Therefore, by introduction of conflict graphs, the problem of channel assignment on a link set is converted into a classical graph coloring problem on the resulting conflict graph.

In the next subsection we recall some useful results on graph coloring, which are used in section IV.

C. Preliminaries on Graph Coloring

Graph coloring problem is one of the most fundamental problems in graph theory. It is known to be NP-hard even for very restricted classes of graphs. A coloring is valid if no two adjacent vertices receive same colors. The minimum number that is needed for a valid vertex coloring for a graph $G(V, E)$ is known as the *chromatic number*, denoted $\chi(G)$.

There are some known upper bounds for $\chi(G)$. The most straightforward one is $\Delta(G) + 1$ [15]:

Proposition 1: $\chi(G) \leq \Delta(G) + 1$.

Here $\Delta(G)$ is the largest degree among G's vertices. Formally,

$$\Delta(G) = \max_{v \in G} Degree(v). \tag{1}$$

Another upper bound for $\chi(G)$ is $\delta(G) + 1$ [17]:

Proposition 2: $\chi(G) \leq \delta(G) + 1$.

Here $\delta(G)$ is the largest $d > 0$ such that G contains a subgraph H in which each node has a degree at least d. For a formal definition, first define

$$LD(G) = \min_{v \in G} Degree(v). \tag{2}$$

Then $\delta(G)$ is defined as

$$\delta(G) = \max_{H \subseteq G} LD(H). \tag{3}$$

Moreover, given a graph $G(V, E)$, there exist efficient algorithms that can compute a valid coloring using at most $\delta(G) + 1$ colors, within $O(|V| + |E|)$ time [7].

IV. Bounds on Number of Channels

In this section we come to the upper bound. By introducing the conflict graph model, the link channel assignment problem is equivalent to the problem of coloring the conflict graph. First we give a trivial upper bound.

Theorem 1: Given N nodes with density D on the plane, for any valid link set LS derived from N, the corresponding conflict graph $G(LS)$ can be colored using $2D - 3$ colors.

Proof: By proposition 1, it suffices to prove that $\Delta(G) \leq 2D - 4$. For any link $PQ \in LS$, we prove that there are at most $2D - 4$ links that conflict with PQ.

Given PQ, we draw a circle centered at P with radius r_i, and a circle centered at Q with radius r_i (see Figure 1). The number of nodes in either circle is at most D. Other than nodes P and Q, there are at most $D-2$ nodes in either circle. Combining the area of two circles, the grey area in Figure 1 contains no more than $2D - 4$ nodes other than nodes P and Q. Since a link which conflicts with link PQ must contain at least one node that is in the grey area, there are at most $2D - 4$ links that conflict with PQ. ■

Note that theorem 1 is valid for *any* settings of interference ratio r. If r is larger than 1, we may get tighter upper bounds. In particular, we have:

Theorem 2: Let $r = 2$. Given N nodes with density D on the plane, for any valid link set LS derived from N, the corresponding conflict graph G can be colored using $1.5D$ colors.

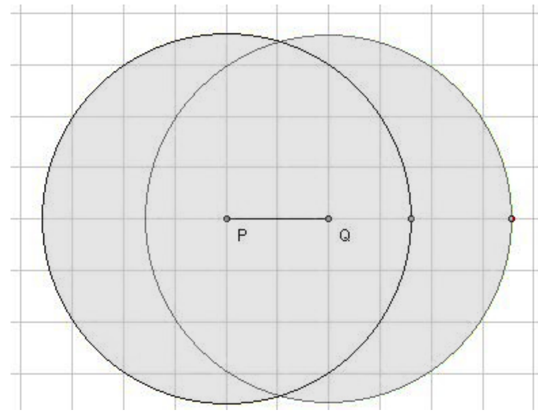

Fig. 1.

Proof: Without loss of generality, assume $r_i = 2$ and $r_t = 1$.

By proposition 2, it suffices to prove that $\delta(G) \leq 1.5D - 1$. From the definition of $\delta(G)$, this is equivalent to

$$\max_{H \subseteq G} \min_{v \in G} Degree(v) \leq 1.5D - 1. \tag{4}$$

To establish (4), it is enough to show that every subgraph of G have a node with degree at most $1.5D - 1$. In other words, we prove that for every subset of link set LS (which will induce a subgraph $H \subseteq G$ in the conflict graph), there exists a link PQ such that there are at most $1.5D - 1$ links that conflict with PQ.

Given a finite subset of links on the plane, there must exist an *edge*, i.e., a straight line, such that at least one node is on the line, and all the nodes are on the right side of the line (see figure 2). We denote the node that is on the line with P, and the link on node P with PQ. In the following we prove that there are at most $1.5D - 1$ links that conflict with PQ.

Given PQ, we can draw a circle centered at P with radius $r_i + r_t = 3$. For every link RS that conflicts with P, at least one node of that link (suppose it is R) should be in P's interference range, so $|PR| \leq 2$. Note also $|RS| \leq 1$, so $|PS| \leq 3$. Therefore, if we draw a circle that is centered at P with radius 3, both Q and R will be in this circle. Similarly we can draw a circle centered at Q with radius 3. Now any link that conflicts with link PQ will be totally contained in the union of the two circles. Since there is no node on the left side of the edge, all links should be on the right side of the edge, as shown by the grey area in Figure 3.

We claim that this grey area can be covered by three circles with radius r_i (*i.e.*, 2). We can put the three circles as follows. Denote the uppermost point on the edge that intersects with

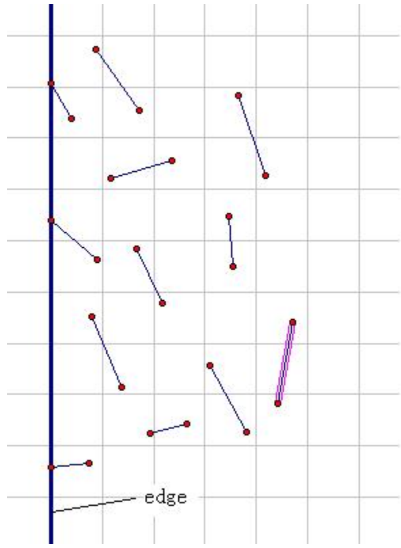

Fig. 2.

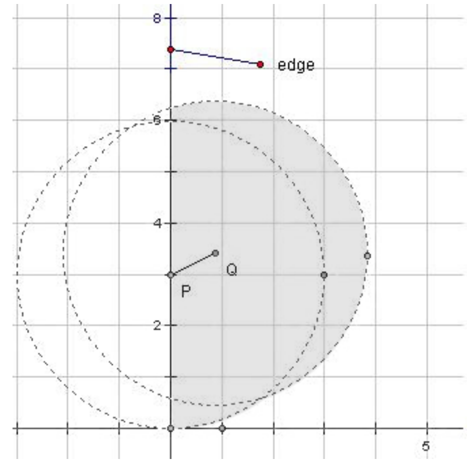

Fig. 3.

the grey area with A (see Figure 4), and the lowermost point on the edge that intersects with the grey area with B, and the mid-point of AB with C. We can put one circle with radius r_i on the plane such that it passes through A and C, and another one circle with radius r_i that passes through C and B. The remaining is to put the third circle to cover the rest of the grey part. There are two cases. Let's denote with C_P the circle centered at P with radius 3, and with C_Q the circle centered at Q with radius 3. In the first case, the rest of the grey part is totally included by C_Q (see figure 4). One may verify that in this case, the points in the rest of the grey part with the largest distance are located on circle C_Q. In figure 4, they are points R and S. By calculating the coordinates of R and S, it turns out that $|RS| < 4$. By putting the third circle with radius 2 centered at the mid-point of RS, we can totally cover the rest of the grey part. In the second case, the rest of the grey part is included by C_P and C_Q, but either C_P or C_Q cannot cover the grey part (see Figure 5). In this case, the points in the rest of the grey part with the largest distance are located on circles C_Q and C_P, respectively. In figure 5, they are points R and S. Similarly, $|RS| < 4$. Therefore the covering is valid.

Since the grey area can be covered by three circles with radius r_i, the number of nodes in this area is at most $3D$. This means there are at most $1.5D$ links in this area. Other than link PQ, there are at most $1.5D - 1$ links that conflict with PQ. So we have proved Theorem 2. ■

Note that the proof for $r = 2$ case in Theorem 2 also works for any $r > 2$. More generally, we have

Theorem 3: If an upper bound U is valid for interference ratio $r = r_1$, and $r_2 > r_1$, then the upper bound U is also valid for $r = r_2$.

Proof: We can assume that the interference range is fixed at $r_i = 1$ in both cases.

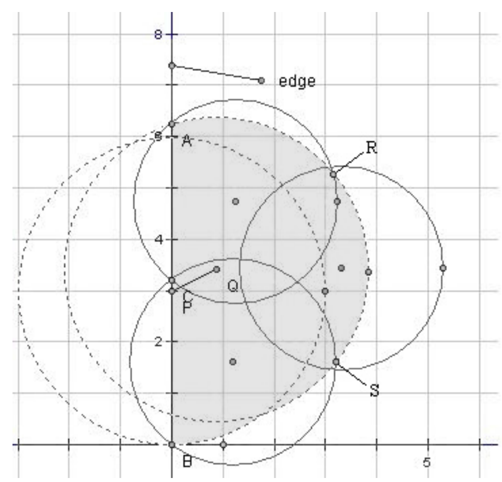

Fig. 4.

In the first case $r = r_1$, this means the transmission range in the first case $r_{t1} = \frac{r_i}{r_1} = \frac{1}{r_1}$ and U is a valid upper bound.

Now in the second case $r = r_2$, this means the transmission range in the second case $r_{t2} = \frac{r_i}{r_2} = \frac{1}{r_2}$. Since $r_2 > r_1$, we have $r_{t1} > r_{t2}$.

This means that the transmission range in the first case ($r = r_1$) is larger than the transmission range in the second case ($r = r_2$). That is to say *any valid link set LS in the second case is also valid in the first case*. Since interference ranges are equal in the two cases, link set LS will result in the same conflict graph in the two cases. Recall the assumption that when $r = r_1$, U colors are enough to satisfy $G(LS)$. Therefore, U is also a valid upper bound when $r = r_2$. ■

Theorem 3 tells us the upper bound is monotonically non-increasing as interference ratio r increases. Moreover, it is

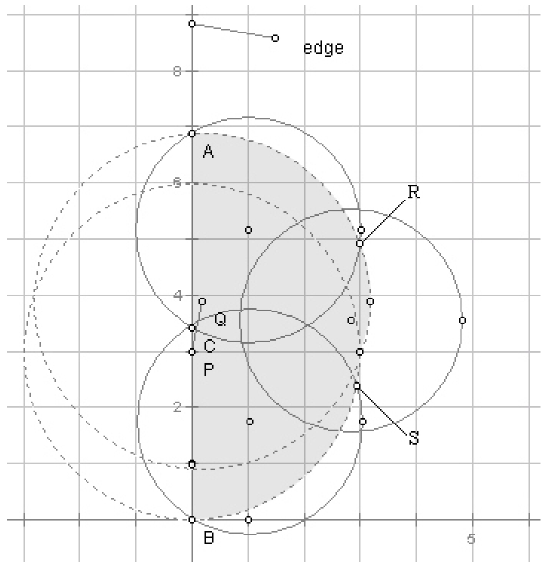

Fig. 5.

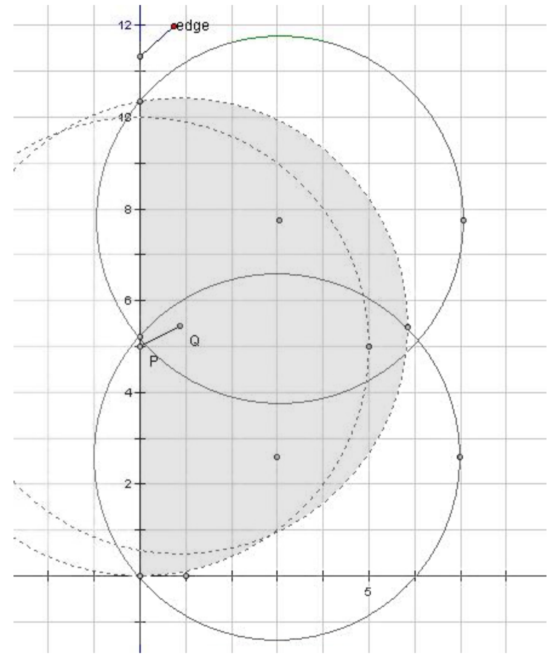

Fig. 6.

intuitive that with larger values of r, the upper bound U can be lowered. This is true when we come to $r = 4$. Using the same technique as in theorem 2, we can prove that

Theorem 4: Let $r = 4$. Given N nodes with density D on the plane, for any valid link set LS derived from N, the corresponding conflict graph G can be colored using D colors.

Proof: The proof is quite similar to that of theorem 2. Without loss of generality, assume $r_i = 4$ and $r_t = 1$.

By proposition 2, it suffices to prove that $\delta(G) \leq D - 1$. To establish this, it is enough to show that every subgraph of G have a node with degree at most $D-1$. In other words, we prove that for every subset of link set LS, there exists a link PQ such that there are at most $D - 1$ links that conflict with PQ.

Similar to the proof of theorem 2, we first pick up an edge, then denote the node on the edge with P, and denote the node on the other end of the link with Q.

Given PQ, we can draw a circle centered at P with radius $r_i + r_t = 5$. For every link RS that conflicts with P, both Q and R will be in this circle. Similarly we can draw a circle centered at Q with radius 5. Now any link that conflicts with link PQ will be totally contained in the union of the two circles. Since there is no node on the left side of the edge, all links should be on the right side of the edge, as shown by the grey area in Figure 6.

Similar to the proof of theorem 2, we claim that this grey area can be covered by two circles with radius r_i (*i.e.*, 4). We can put the three circles as follows. Denote the uppermost point on the edge that intersects with the grey area with A (see Figure 6), and the lowermost point on the edge that intersects with the grey area with B, and the mid-point of AB with C. We can put one circle with radius r_i on the plane such that it passes through A and C, and another one circle with radius r_i that passes through C and B. Figure 6 illustrates the covering.

Since the grey area can be covered by two circles with radius r_i, the number of nodes in this area is at most $2D$. This means there are at most D links in this area. Other than link PQ, there are at most $D - 1$ links that conflict with PQ. So we have proved Theorem 4. ∎

V. Lower Constraints on The Upper Bound

In this section we construct and analyze some specific transmission scenarios, and try to examine how far the upper bound could be reduced.

Theorem 5: When $r = 1$, the upper bound cannot be reduced to lower than $D - 1$.

Proof: We construct an example as in Figure 7. Given density D (in Figure 7 we take $D = 13$ as an example), first draw a circle with radius $r_i/2$ and put $D - 1$ nodes equally dispersed on the circle. For each node on the circle, draw a link with length r_t directed against the center of the circle, as shown in Figure 7. Here $r_i = r_t = 4$.

One may verify that the constructed node set above is of density D. One can also verify that for the above set of links, the corresponding conflict graph is a $(D - 1)$-clique, which needs exactly $D-1$ colors to color. So the upper bound cannot be reduced to lower than $D - 1$. ∎

Theorem 6: When $r = 2$, the upper bound cannot be reduced to lower than $\frac{2}{3}(D - 1)$.

Proof: Similar to the proof of theorem 5, we prove theorem 6 using an example in Figure 8. We set odd number of nodes equally dispersed on a circle (in Figure 8 the number is 9). To guarantee that any two links conflict with each other,

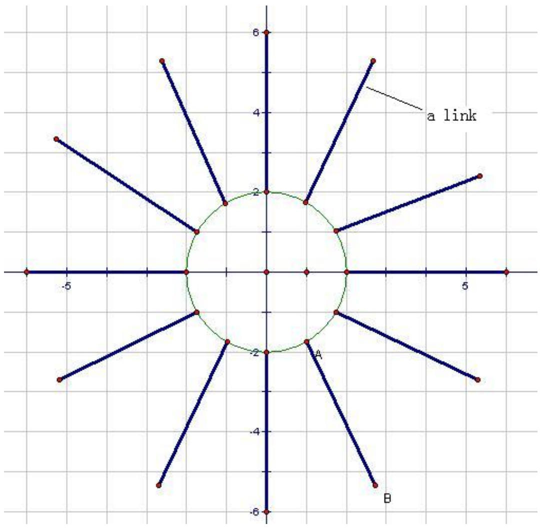

Fig. 7. Every bold line segment represents a link.

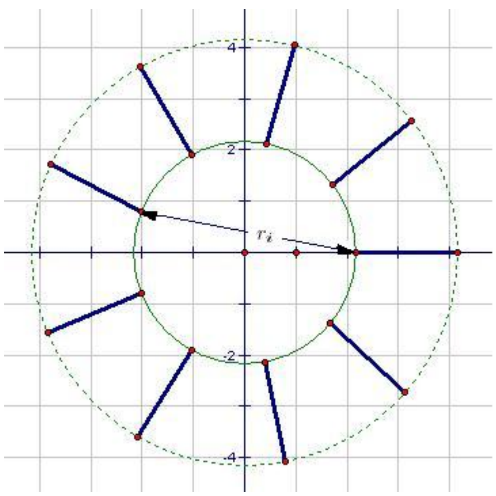

Fig. 8.

we set the two nodes which have the largest distance between them have distance exactly r_i. This makes the inner circle have a radius slightly larger than $r_i/2$. Then for each node on the circle, draw a link with length r_t directed against the center of the circle. Since $r_t = r_i/2$, the other ends of these links are on an outer circle with radius slightly larger than r_i.

Suppose the number of links in this example is n (n is an odd number). Since the outer circle is slightly larger than a circle with radius r_i, a circle with radius r_i can cover at most $\lceil n/2 \rceil$ nodes on the outer circle, plus n nodes on the inner circle. So for this example, $D \leq n + \lceil n/2 \rceil \leq \frac{3}{2}n + 1$. That is $n \geq \frac{2}{3}(D-1)$. Note also that this link set need n colors to color. So at least $\frac{2}{3}(D-1)$ colors are needed. ■

Theorem 7: The upper bound cannot be reduced to lower than $0.5D$, for any r.

r	Upper bound	Upper bound must be equal or greater than
1	$2D-3$	$D-1$
2	1.5D	$\frac{2}{3}(D-1)$
4	D	$0.5D$
∞	$0.5D$	$0.5D$

TABLE I

A SUMMARIZATION OF THE RESULTS

Proof: This theorem is proved by the following example. Put a set of D nodes that are located almost on the same point. In other words, each two nodes are within distance ϵ, where ϵ is a positive number that is small enough. The links are formed by connecting any two nodes. Then it is clear that there are $0.5D$ links that conflict each other, and the node density is exactly D. Thus the number of channels needed in this case is just $0.5D$. ■

VI. SUMMARY AND DISCUSSION

First we summarize our results in table I. Note that for $r \to \infty$, suppose $r_i = 1$ is fixed, then $r_t \to 0$. This means all links are having length 0, so any link will conflict with at most $\frac{D}{2} - 1$ links, yielding the exact $0.5D$ bound.

From the results, we recognize that the number of channels needed for a conflict-free communication scales linearly with the node density D, and is non-increasing as the interference ratio r increases.

Note that although in table I there are upper bounds for only several values of r, by applying theorem 3 we can actually get valid upper bounds for any values of r. For example, for $r = 3$, the bound $1.5D$ is still valid. An interesting question is whether the upper bound can be expressed as a continuous function of r.

We observe that there is still a gap between the provable upper bound and its lower constraint. This means that the upper bound is possible to be further improved. However, this gap seems very difficult to be fully eliminated. To understand the difficulty of this problem, we compare our problem with the *imperfection ratio problem* in graph theory. The imperfection ratio of graph G is defined as the supreme of the ratio between its chromatic number and its clique number. The imperfection ratio on a *unit disk graph*, which is similar to the resulting conflict graph in our problem, is upper-bounded by 2.155 [5], and lower-bounded by 1.5. There is a gap between the two known bounds. Since the clique number can be regarded as an alternative definition of density, there is some similarity between our problem and the imperfection ratio problem. This comparison, though by no means formal, may give us some understanding on the difficulty of overcoming the gap in our problem.

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